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Multipole expansion method for supernova neutrino oscillations

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Abstract. We develop a multipole expansion method to calculate collective neutrino oscillations in supernovae using the neutrino bulb model. We test this method with two representative neutrino energy spectra, and we find that this method can be much more efficient than the angle-bin method previously used in the literature. The multipole expansion method also provides interesting insights into multi-angle calculations that were accomplished previously by the angle-bin method.

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1 Introduction

It is well known that neutrino oscillation probabilities can be modified by ambient matter through the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1, 2]. Likewise, neutrino oscillations can also be affected by the presence of ambient neutrinos [3, 4]. The phenomenon of collective neutrino oscillations (e.g., [5–10]; see [11] for a recent but incomplete review), caused by neutrino self-interaction via the Z -boson mediation, continues to surprise us with new features and instabilities (e.g., [12–21]).

Unlike the MSW effect, collective neutrino flavor transformation caused by neutrino self-interaction couples the quantum states of neutrinos themselves and, therefore, is nonlinear. Except for a few very simplistic models (see, e.g., [8, 22, 23]), the flavor evolution of a dense neutrino gas can be solved only through numerical methods. This non-linearity together with the inhomogeneous and anisotropic physical environment makes it a daunting task to solve collective neutrino oscillations in a (core-collapse) supernova even numerically. The most sophisticated calculations of this kind so far have adopted the “neutrino bulb model” [7] which assumes spherical symmetry for the supernova environment. The current approach of solving neutrino oscillations in the neutrino bulb model is to discretize the neutrino emission (zenith) angle as well as the neutrino energy and solve millions to tens of millions of coupled differential nonlinear equations simultaneously. Although this “angle-bin method” is straightforward to implement, a severe drawback of this approach is that a large number of angle bins ($\gtrsim 1000$) are required to achieve numerical convergence even in the regime where no significant neutrino oscillation is observed [7, 24].

Because of its spherical symmetry, the neutrino bulb model does not completely capture the complexity of the supernova environment revealed in recent multi-dimensional supernova simulations (e.g., [25–27]). Even if the supernova is approximately spherically symmetric, the spherical symmetry in neutrino oscillations can be broken spontaneously because of the vector-vector coupling nature of the neutrino self-interaction [19, 28, 29]. In the cases where the spherical symmetry is indeed maintained, the current technique is still insufficient for solving neutrino oscillations in the presence of the neutrino halo [17] and/or transition magnetic moments of Majorana neutrinos [20, 21]. It is clear that a new computing model or models are necessary to address the above challenges. Because the new model(s) will be more complicated and computationally more demanding than the neutrino bulb model, it is also clear that the angle-bin method needs to be replaced by a more efficient approach.

A multipole expansion approach using Legendre polynomials was employed in an earlier study with the focus on the “kinematical decoherence” of a homogeneous neutrino gas [30]. A different moment expansion method was later proposed for the neutrino bulb model [31].

In this paper we develop a multipole expansion method similar to that in [30] for the neutrino bulb model. The rest of the paper is organized as follows. In Section 2 we explain the formalism of the new multipole expansion method. In Section 3 we test the multipole expansion method with two representative neutrino energy spectra. We also discuss the efficiency of this method, some physical insights behind it, and how it may be improved. In Section 4 we give our conclusions. Some of the approximations used in our approach are listed in Appendix A.

2 Formalism

Here we consider two-flavor neutrino oscillations (i.e. $\nu_e \leftrightarrow \nu_x$ and $\bar{\nu}_e \leftrightarrow \bar{\nu}_x$) in the neutrino bulb model. Generalization to three neutrino flavors is straightforward. The neutrino bulb model assumes spherical symmetry about the center point of the supernova and azimuthal symmetry about any radial direction from this center point. In this model the flavor density matrix $\rho_{E,u}(r)$ of the neutrino at radius r depends only on the energy E and trajectory $u = \cos(2\vartheta_0)$ of the neutrino, where the (zenith) emission angle ϑ_0 is defined with respect to the normal of the neutrino sphere at radius R . We adopt the convention of the neutrino flavor isospin [6] so that the diagonal elements of $\rho_{E,u}$ (in flavor basis) with $E < 0$ are proportional to the negative number densities of $\bar{\nu}_e$ and $\bar{\nu}_x$ with energy $|E|$. The flavor density matrices of neutrinos and anti-neutrinos are normalized in the following manner:

$$\text{Tr} \int_0^\infty \rho_{E,u} dE = 1, \quad \text{Tr} \int_{-\infty}^0 \rho_{E,u} dE = -\frac{\Phi_{\bar{\nu}}}{\Phi_{\nu}}, \quad (2.1)$$

where Φ_{ν} and $\Phi_{\bar{\nu}}$ are the total number fluxes of the neutrino and anti-neutrino, respectively.

The equation of motion for density matrix $\rho_{E,u}$ is

$$iv_u \partial_r \rho_{E,u} = [\mathbf{H}_{E,u}, \rho_{E,u}], \quad (2.2)$$

where ∂_r is differentiation with respect to radius r ,

$$v_u = \sqrt{1 - \left(\frac{R}{r}\right)^2 \sin^2 \vartheta_0} = \sqrt{1 - \left(\frac{R}{r}\right)^2 \left(\frac{1-u}{2}\right)} \quad (2.3)$$

is the radial component of the neutrino velocity, and

$$\begin{aligned} \mathbf{H}_{E,u} &= \mathbf{H}_{\text{vac}} + \mathbf{H}_{\text{matt}} + \mathbf{H}_{\nu\nu} \\ &= \frac{\mathbf{M}^2}{2E} + \sqrt{2}G_{\text{F}}\mathbf{L} + \frac{\sqrt{2}G_{\text{F}}\Phi_{\nu}}{2\pi R^2} \int_{\sqrt{1-(R/r)^2}}^1 dv_{u'} \int_{-\infty}^{\infty} dE' (1 - v_u v_{u'}) \rho_{E',u'} \end{aligned} \quad (2.4)$$

is the Hamiltonian. In the above equation \mathbf{M}^2 is the neutrino mass-squared matrix in flavor basis, G_{F} is the Fermi coupling constant, and \mathbf{L} is the matrix of net charged-lepton number densities. Here, for simplicity, we have assumed that neutrinos are emitted isotropically from the neutrino sphere.

We define the n 'th multipole/moment of the neutrino flavor matrix to be

$$\rho_{E,n} = \int_{-1}^1 \rho_{E,u} P_n(u) du, \quad (2.5)$$

where $P_n(u)$ are the standard Legendre polynomials. The reason that we use $u = \cos(2\theta_0)$ instead of v_u (as in [30, 31]) in the definition of the multipoles is that u has a fixed range $[-1, 1]$ but the range of v_u is $[1 - \sqrt{1 - (R/r)^2}, 1]$ which changes with r . Using the normalization condition

$$\int_{-1}^1 P_m(u) P_n(u) du = \frac{2}{2n+1} \delta_{mn} \quad (2.6)$$

we obtain

$$\rho_{E,u} = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \rho_{E,n} P_n(u). \quad (2.7)$$

Instead of transforming eq. (2.2) to multipole basis directly, we note that (multi-angle) neutrino oscillations in the neutrino bulb model usually occur at

$$z \equiv \frac{R^2}{4r^2} \ll 1. \quad (2.8)$$

Therefore, we first expand eq. (2.2) in terms of z as in [12, 32]. Using the appropriate lowest order approximation for each term in eq. (2.4) we obtain (see Appendix A)

$$i\partial_r \rho_{E,u} \approx [\Omega - u\lambda\sigma_3 + \mu(2-u)\rho_0 - \mu\rho_1, \rho_{E,u}], \quad (2.9)$$

where

$$\Omega = \frac{\omega}{2} \begin{bmatrix} -\cos 2\theta_{\text{eff}} & \sin 2\theta_{\text{eff}} \\ \sin 2\theta_{\text{eff}} & \cos 2\theta_{\text{eff}} \end{bmatrix}, \quad (2.10)$$

$$\lambda(r) = \frac{G_F n_e(r) z(r)}{\sqrt{2}} \quad (2.11)$$

with n_e being the net electron number density, σ_3 is the third Pauli matrix,

$$\mu(r) = \frac{\sqrt{2} G_F \Phi_\nu z^2(r)}{2\pi R^2}, \quad (2.12)$$

and

$$\rho_n(r) = \int_{-\infty}^{\infty} \rho_{E,n}(r) dE. \quad (2.13)$$

In eq. (2.10), $\omega \equiv \Delta m^2/2E$ is the vacuum oscillation frequency of the neutrino with Δm^2 being the mass-squared difference of neutrino, and $\theta_{\text{eff}} \ll 1$ is the effective mixing angle of neutrino in matter [6, 8].

Using identity

$$\int_{-1}^1 P_m(u) P_n(u) u du = \frac{2(n+1)}{(2n+1)(2n+3)} \delta_{n+1,m} + \frac{2n}{(2n-1)(2n+1)} \delta_{n-1,m} \quad (2.14)$$

	SS spectra	MS spectra
L_{ν_e} (10^{51} ergs/sec)	1.0	4.1
$L_{\bar{\nu}_e}$ (10^{51} ergs/sec)	1.0	4.3
$L_{\nu_x/\bar{\nu}_x}$ (10^{51} ergs/sec)	1.0	7.9
T_{ν_e} (MeV)	2.8	2.1
$T_{\bar{\nu}_e}$ (MeV)	4.0	3.4
$T_{\nu_x/\bar{\nu}_x}$ (MeV)	6.3	4.4
η_{ν_e}	3.0	3.9
$\eta_{\bar{\nu}_e}$	3.0	2.3
$\eta_{\nu_x/\bar{\nu}_x}$	3.0	2.1

Table 1. The two sets of neutrino luminosities and spectral parameters used in our calculations. They are adapted from [7] and [13], respectively, and are known to produce a single swap/split (SS) and multiple swaps/splits (MS) in final neutrino fluxes.

we rewrite eq. (2.9) in the multipole basis,

$$i\partial_r \rho_{E,n} \approx [\Omega + \mu(2\rho_0 - \rho_1), \rho_{E,n}] - [\lambda\sigma_3 + \mu\rho_0, a_n \rho_{E,n+1} + b_n \rho_{E,n-1}], \quad (2.15)$$

where

$$a_n = \frac{n+1}{2n+1}, \quad b_n = \frac{n}{2n+1}. \quad (2.16)$$

3 Validation and discussion

To validate and show the usefulness of the multipole expansion method we solve eq. (2.15) numerically (using the multipole expansion method) with two sets of initial neutrino energy spectra. We assume that the matter density is not large enough to suppress collective neutrino oscillations, and we take $\lambda = 0$. For comparison we also solve eq. (2.2) numerically (using the angle-bin method) with replacement

$$\mathbf{H}_{\text{vac}} + \mathbf{H}_{\text{matt}} \longrightarrow \Omega.$$

In both kinds of calculations we use mass-squared difference $\Delta m^2 = -3 \times 10^{-3} \text{ eV}^2$ (i.e. with the inverted neutrino mass hierarchy), effective mixing angle $\theta_{\text{eff}} = 0.01$, 100 energy bins per neutrino flavor, and the radius of the neutrino sphere $R = 11 \text{ km}$.

The two sets of neutrino spectra used in our calculations are adapted from [7] and [13], respectively, which are known to produce a single swap/split (SS) and multiple swaps/splits (MS) in final neutrino fluxes. In both sets the neutrino spectra are described by the Fermi-Dirac distribution

$$f_\nu(E) \propto \frac{E^2}{1 + \exp\left(\frac{E}{T_\nu} - \eta_\nu\right)}, \quad (\nu = \nu_e, \bar{\nu}_e, \nu_x, \bar{\nu}_x) \quad (3.1)$$

with the parameters listed in table 1.

In figure 1 we show the neutrino fluxes at 200 km computed in multipole basis with 25 mutipoles (i.e. with $\rho_{E,n \geq 25} = 0$) and with the SS spectra. In the same figure we also plot

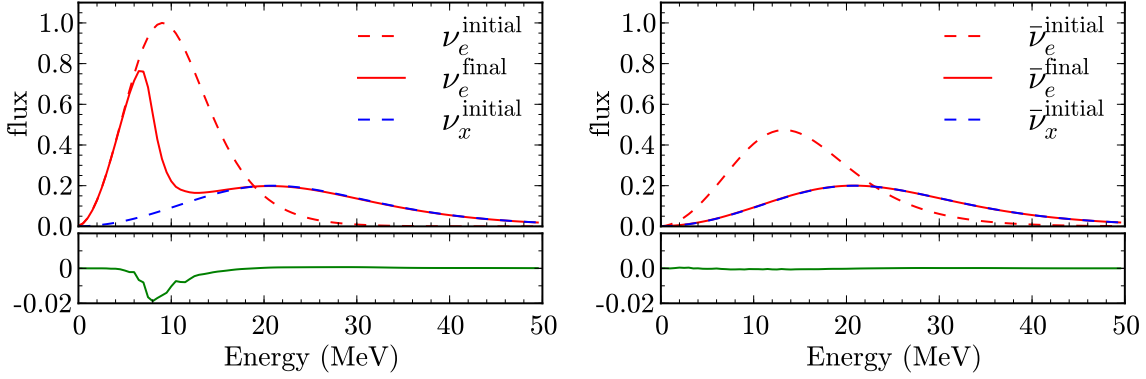


Figure 1. Neutrino fluxes (in arbitrary units) in the SS spectrum case. The left and right panels are for neutrino and anti-neutrino, respectively. In the top panels the dashed curves are for the initial spectra at the neutrino sphere, and the solid curves are for the fluxes at 200 km computed in multipole basis with 25 multipoles. The bottom panels show the differences between the final ν_e fluxes computed in multipole basis and angle basis (with 1200 angle bins).

the differences between these results and those computed in angle basis with 1200 angle bins. One can see that these two calculations agree with each other very well. We note that the small differences between the two results are partly due to the approximations we made in eq. (2.9) which is employed in the multipole expansion method.

To understand why so few multipoles are needed, we consider the strengths of the multipoles defined as follows,

$$S_{E,n} \equiv \left[(2n+1) \frac{\text{Tr} \rho_{E,n}^2}{(\text{Tr} \rho_{E,0})^2} \right]^{1/2}. \quad (3.2)$$

In particular, $S_{E,0}$ is an indicator of the flavor polarization of the angle averaged neutrino flux with energy E , and $S_{E,0} = 1/\sqrt{2}$ implies complete flavor depolarization, i.e. equal number of ν_e ($\bar{\nu}_e$) and ν_x ($\bar{\nu}_x$). Because we consider only the coherent forward scattering of neutrinos outside the neutrino sphere, $\text{Tr} \rho_{E,u}$ and, therefore, $\text{Tr} \rho_{E,n}$, do not change with radius. Furthermore, because we do not consider the quantum decoherence of the neutrino states, $\text{Tr} \rho_{E,u}^2$ are also constant. Using eqs. (2.6) and (2.7) it is straightforward to show that

$$\frac{d}{dr} \sum_n S_{E,n}^2 = 0. \quad (3.3)$$

In figure 2 we show how

$$S_n \equiv \left(\int_0^\infty S_{E,n}^2 \text{Tr} \rho_{E,0} dE \right)^{1/2}, \quad \bar{S}_n \equiv \left(\int_{-\infty}^0 S_{E,n}^2 |\text{Tr} \rho_{E,0}| dE \right)^{1/2} \quad (3.4)$$

evolve as functions of radius. One can see that, right after collective neutrino oscillations begin, both S_n and \bar{S}_n grow exponentially with radius. This result suggests that all multipoles become unstable simultaneously which one may expect from the ansatz of the linear stability analysis in angle basis [32]. However, this result seems to be contrary to a previous study of the homogeneous gas of neutrinos where higher multipoles are populated successively by diffusion from lower multipoles [30].

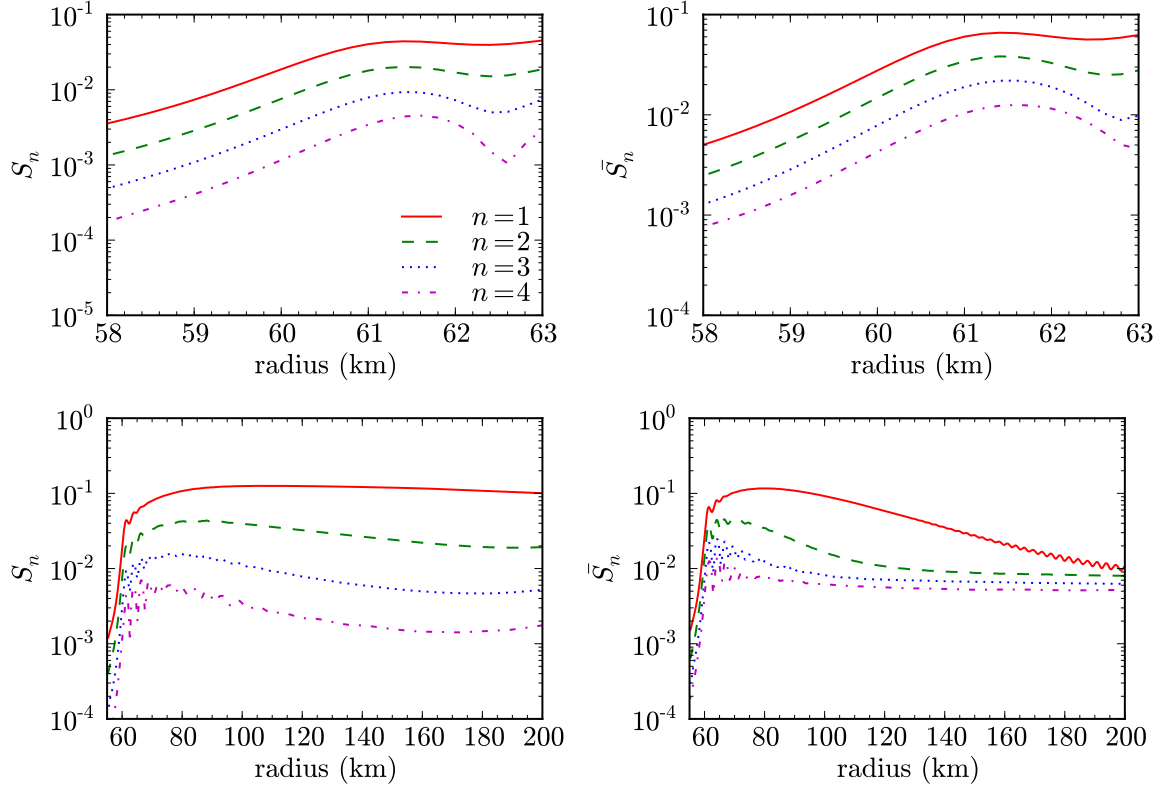


Figure 2. The overall strengths of the multipoles of the neutrino fluxes as functions of radius in the SS spectrum case. The left and right panels are for neutrino and anti-neutrino, respectively. The top panels show the exponential growth of S_n and \bar{S}_n right after collective oscillations begin. The bottom panels show the behavior of these multipole strengths in the whole regime where collective oscillations are active.

From figure 2 one can also see that S_n and \bar{S}_n are never large for $n > 0$. This property together with eq. (3.3) implies that $S_{E,0}$ is almost constant and that there is no significant flavor depolarization in this particular case. This result is, of course, already known from figure 3 in [33], which motivated us to solve neutrino oscillations in multipole basis in the first place. From figure 2 one may conclude that only the first very few multipoles, far fewer than 25, are needed for this calculation. This is indeed true but only up to a certain radius after which spurious oscillations of big amplitudes would occur. This phenomenon implies that some kind of diffusion among multipoles (as suggested in [30]) may indeed exist while collective oscillations are active. Such spurious oscillations may be suppressed by implementing an appropriate closure scheme (as, e.g., in [34]). We note that any closure that insures the accuracy of the first multipoles would be sufficient for most physical applications at $r \gg R$.

It is interesting to note that, in the regime where no significant neutrino oscillations occur, all multipoles with $n > 0$ are very small. This is because in this regime all neutrinos essentially stay in the same flavor states as they are at the neutrino sphere. Therefore, one can set all but the first two multipoles to 0 initially and solve eq. (2.15) for $\rho_{E,0}$ and $\rho_{E,1}$ only until the magnitudes of $\rho_{E,1}$ cross certain threshold which depends on the desired precision.¹

¹In the case where the angle distributions of the neutrino fluxes are not isotropic on the neutrino sphere, e.g. in [35], one needs to solve for the first several multipoles before collective neutrino oscillations begin.

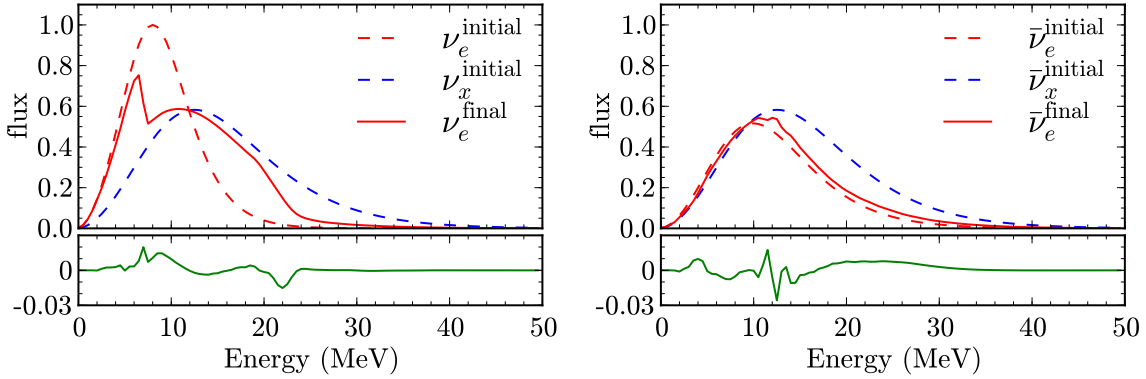


Figure 3. Neutrino fluxes at 400 km for the MS spectrum case. The notations are the same as in figure 1. The computations was performed with 300 multipoles in multipole basis and 50000 angle bins in angle basis.

(Solving eq. (2.15) for $\rho_{E,0}$ only is equivalent a single angle approximation, which may lead to wrong results [16].) More moments can be added adaptively from this point on as necessary.

In contrast, the angle-bin method requires a large number of angle bins ($N_a \gtrsim 1000$) to achieve numerical convergence even in the regime where no physical oscillations occur [7]. It was shown in [24] that the numerical solutions to eq. (2.9) with discrete angle bins have flavor instabilities that are absent in the continuum limit $N_a \rightarrow \infty$. These spurious flavor instabilities can be suppressed by employing a large number of angle bins. Our calculation shows that such spurious oscillations do not occur, at least in this particular case, in the numerical solution to eq. (2.15) which is completely equivalent to eq. (2.9).

We also performed the calculations for the MS spectrum. It was reported in [13] that one needed 15000 angle bins to achieve “apparent convergence” for this case. We found that true convergence could be achieved with 50000 angle bins in our calculation. In comparison only 300 multipoles are required to achieve numerical convergence in multipole basis. In figure 3 we show the neutrino fluxes at 400 km in these calculations, and in figure 4 we show the multipole strengths S_n and \bar{S}_n as functions of radius.

Our discussions for the SS spectrum case also apply to MS spectrum case. However, we note that the strengths of the multipoles in this case do not decrease rapidly with increasing n . The first several multipoles oscillate with radius when collective oscillations begin, and their strengths are of similar magnitude. Significant more multipoles are required to achieve numerical convergence in the MS spectrum case than in the SS spectrum case.

4 Conclusion

We have developed a multipole expansion method for the multi-angle calculations of collective neutrino oscillations in supernovae. We have tested this method with two representative neutrino energy spectra. Our calculations show that, at least for these two cases, the multipole expansion method requires solving far fewer number of equations (by approximately two orders of magnitude) than the traditional angle-bin method does.

The relative efficiency of the multipole expansion method are twofold: (a) Very few multipoles are needed in the regime where no physical oscillations occur. In contrast, a large number of angle bins are required to suppress spurious oscillations in the angle-bin

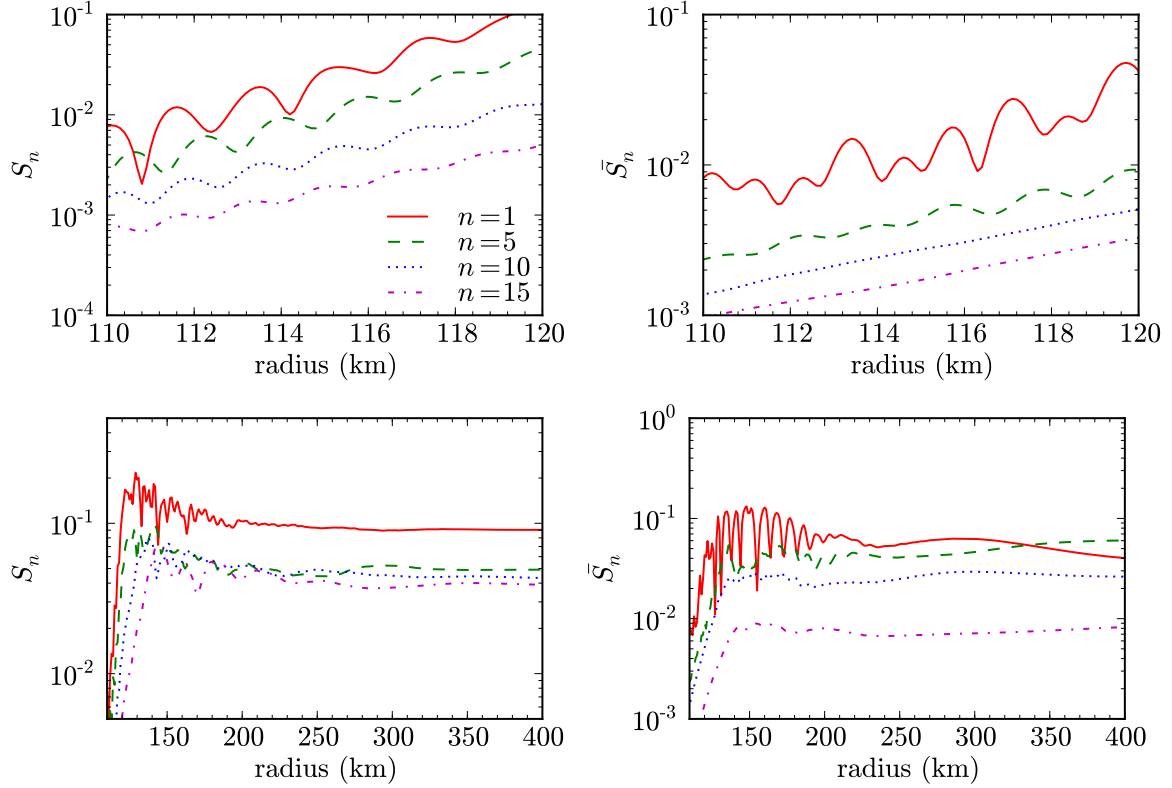


Figure 4. Same as figure 2 but for the MS spectrum case.

method. (b) The strengths of the multipoles decrease with increasing multipole index n in the regime where collective neutrino oscillation are active. Although we have tested the multipole expansion method for two representative neutrino energy spectra, the conclusions obtained here may apply to many other situations. More work is needed to be done in this regard.

The multipole expansion method can potentially be adapted to more realistic supernova models. The efficiency of this method may be further improved if an appropriate closure scheme is employed.

A Approximations

Like in previous works (e.g., [12, 32]) we expand the Hamiltonian in eq. (2.4) in terms of $z = R^2/4r^2$ and keep the leading order terms for each part of the Hamiltonian. Higher order terms can be included for more accuracy. For vacuum Hamiltonian we have

$$\frac{\mathbf{H}_{\text{vac}}}{v_u} \xrightarrow{z \ll 1} \frac{\omega}{2} \begin{bmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{bmatrix} + \mathcal{O}(z), \quad (\text{A.1})$$

where $\omega = \Delta m^2/2E$ is the vacuum oscillation frequency with Δm^2 being the neutrino mass-squared difference, and θ_v is the neutrino vacuum mixing angle. We have dropped the trace term which has no impact on neutrino oscillations. We note that the dispersion of $\mathbf{H}_{\text{vac}}/v_u$ in neutrino energy E , which is much larger than its dispersion in angle parameter u , plays

an essential role in collective neutrino oscillations [6]. We also note that the terms of order z or higher are unlikely to be important because H_{vac} is smaller than $H_{\nu\nu}$ where collective oscillations are active.

For matter Hamiltonian we have

$$\frac{H_{\text{matt}}}{v_u} \xrightarrow{z \ll 1} \lambda(r)[z^{-1} + (1-u)] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \mathcal{O}(z^2), \quad (\text{A.2})$$

where

$$\lambda(r) = \frac{G_F n_e(r) z(r)}{\sqrt{2}}, \quad (\text{A.3})$$

and where we have again dropped the trace term. The terms on the right hand side of eq. (A.2) that do not depend on u have no effect on collective neutrino oscillations other than resetting the vacuum mixing angle θ_v to a smaller value, which we denote by θ_{eff} [6, 8]. The angle dependent term in eq. (A.2) can lead to suppression of collective oscillations in the case of very high matter density, which can be important near the neutrino sphere and/or during the accretion phase of the explosion [12]. The terms of order z^2 or higher in eq. (A.2) are unlikely to be important because the matter density itself decreases rapidly with r . In the regime where the matter density is much larger than $G_F^{-1}|\omega|$ for typical neutrino energies, we obtain

$$\frac{1}{v_u}(H_{\text{vac}} + H_{\text{matt}}) \longrightarrow \frac{\omega}{2} \begin{bmatrix} -\cos 2\theta_{\text{eff}} & \sin 2\theta_{\text{eff}} \\ \sin 2\theta_{\text{eff}} & \cos 2\theta_{\text{eff}} \end{bmatrix} - u\lambda(r) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (\text{A.4})$$

For the neutrino self-coupling Hamiltonian we have

$$\frac{H_{\nu\nu}}{v_u} \approx \mu(r)[(2-u)\rho_0 - \rho_1] + \mathcal{O}(z^3), \quad (\text{A.5})$$

where

$$\mu(r) = \frac{\sqrt{2}G_F\Phi_\nu z^2}{2\pi R^2}, \quad (\text{A.6})$$

and

$$\rho_n = \int_{-\infty}^{\infty} \rho_{E,n} dE. \quad (\text{A.7})$$

The angle dependence of $H_{\nu\nu}$ can lead to the multi-angle suppression of collective oscillations [16, 32].

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